Change of Variables

We begin with two familiar changes of variables. Look closely at the integration procedure.

Example 16.7.1. Integrate $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{2-x^2}} dx$

Example 16.7.2. Integrate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} e^{1-x^2-y^2}\, dydx$

Transformations of $\mathbb{R}^2$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a differential multivariable, vector valued function.

We say T is *injective* or *one-to-one* from a region S to a region R if $T(P) = T(Q)$ only when $P = Q$.

Let $T(u, v) = \langle g(u, v), h(u, v) \rangle$ be an injective function from a closed bounded region S to a closed bounded region R. Assume T is differentiable on S. If f is continuous on R, then

$$\iint_R f(x, y)dA = \iint_S f(g(u, v), h(u, v))_____$$

Exercise 16.7.3. Fill in the blank above.

Example 16.7.4. Let R be the closed square with vertices $(0,0), (1,-1), (2,0)$, and $(1,1)$. Evaluate

$$\iint_R \sqrt{\frac{x-y}{x+y+1}}\, dA.$$

1. Find a linear transformation which makes the integrand simpler.

2. Determine the region S in the uv-plane which corresponds to R.

3. Find the Jacobian $J(u,v)$

4. Evaluate the integral after the change of variables.

Example 16.7.5. Let R be the region in the first quadrant bounded by the curves $x = y^2, x = y^2 - 4, x = 9 - y^2$, and $x = 16 - y^2$. Evaluate $\iint_R y^2 \, dA$.

1. What about the integral is difficult??

2. Select an appropriate injective transformation $(x, y) \to (u, v)$. Compute the Jacobian of the transformation after solving for x and y.

3. Evaluate the integral after the change of variables.